Class : I M.Sc., Physics

Course : MATHEMATICAL METHODS FOR PHYSICISTS

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Unit – I –a : Vector Analysis

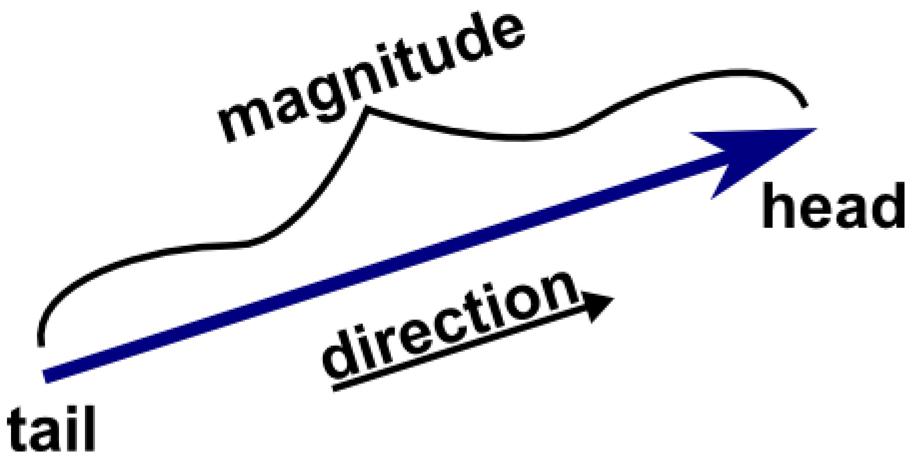
Gradient, divergence, curl, and Laplacian – Orthogonal curvilinear coordinate systems – spherical coordinate system and cylindrical coordinate system– Expression for gradient, divergence, curl and Laplacian

Unit – I –a : Vector Analysis

An introduction to vectors

Definition

A vector is an object that has both a magnitude and a direction. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head.



- Two vectors are the same if they have the same magnitude and direction. This means that if we take a vector and translate it to a new position (without rotating it), then the vector we obtain at the end of this process is the same vector we had in the beginning.

- Two examples of vectors are those that represent force and velocity. Both force and velocity are in a particular direction. The magnitude of the vector would indicate the strength of the force or the speed associated with the velocity.

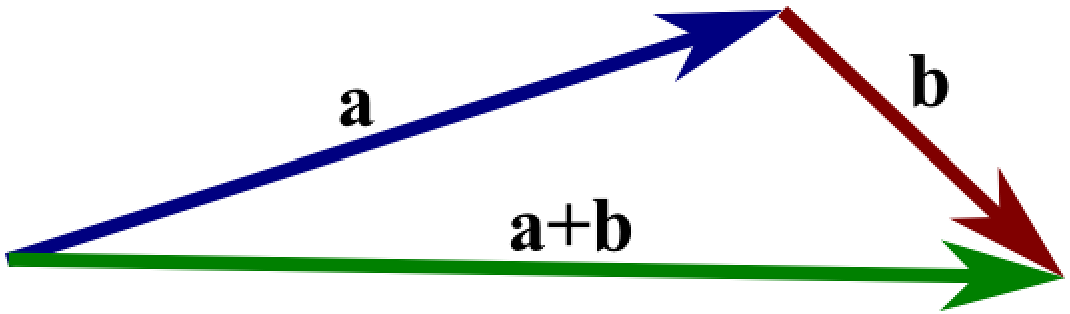
- We denote vectors using boldface as in **a** or **b**. Especially when writing by hand where one cannot easily write in boldface, people will sometimes denote vectors using arrows as in $\vec{a}$ or $\vec{b}$, or they use other markings. We won't need to use arrows here. We denote the magnitude of the vector a by $\|a\|$. When we want to refer to a number and stress that it is not a vector, we can call the number a scalar. We will denote scalars with italics, as in a or b.

Operations on vectors

We can define a number of operations on vectors geometrically without reference to any coordinate system. Here we define addition, subtraction, and multiplication by a scalar.

Addition of vectors:

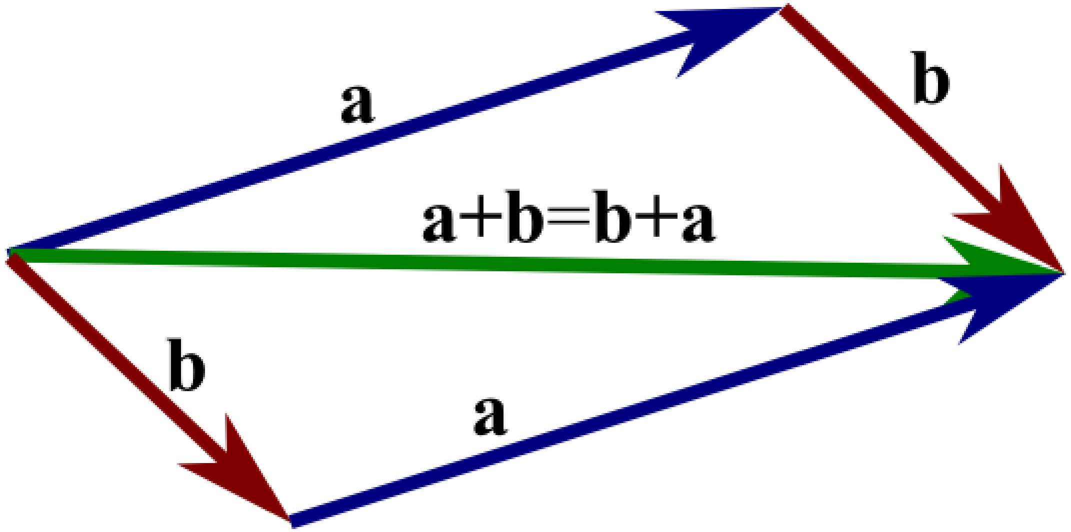
Given two vectors **a** and **b**, we form their sum **a+b**, as follows. We translate the vector **b** until its tail coincides with the head of **a**. (Recall such translation does not change a vector.) Then, the directed line segment from the tail of **a** to the head of **b** is the vector **a+b**



Addition of vectors satisfies two important properties

- The commutative law, which states the order of addition doesn't matter: $a+b = b+a$

This law is also called the parallelogram law, as illustrated in the below image. Two of the edges of the parallelogram define $a+b$, and the other pair of edges define $b+a$. But, both sums are equal to the same diagonal of the parallelogram.

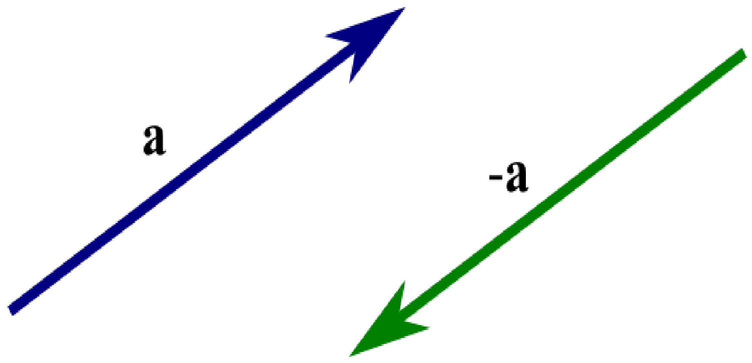


- The associative law, which states that the sum of three vectors does not depend on which pair of vectors is added first:

$$(a+b)+c = a+(b+c)$$

Vector subtraction

- Before we define subtraction, we define the vector **−a**, which is the opposite of **a**. The vector **−a** is the vector with the same magnitude as a but that is pointed in the opposite direction.



- We define subtraction as addition with the opposite of a vector: **b−a = b+(−a)**

Scalar multiplication

- Given a vector **a** and a real number (scalar) λ, we can form the vector **λa** as follows. If λ is positive, then λa is the vector whose direction is the same as the direction of **a** and whose length is λ times the length of **a**. In this case, multiplication by λ simply stretches (if $\lambda>1$) or compresses (if $0<\lambda<1$) the vector **a**.
- If, on the other hand, λ is negative, then we have to take the opposite of a before stretching or compressing it.

Scalar multiplications satisfies many of the same properties as the usual multiplication

- s(a+b) = sa+sb (distributive law, form 1)
- (s+t)a = sa+ta (distributive law, form 2)
- 1a = a
- (−1)a = −a
- 0a = 0

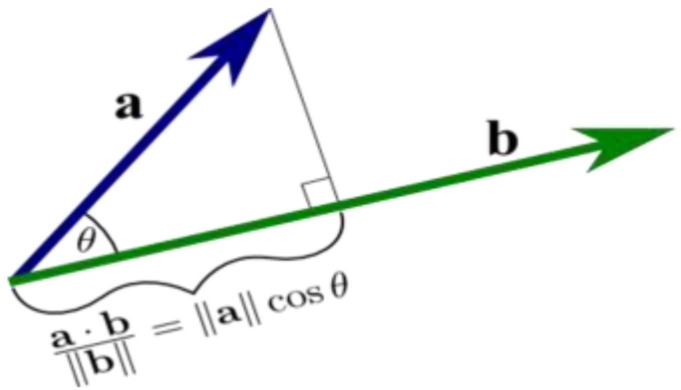
In the last formula, the zero on the left is the number 0, while the zero on the right is the vector 0, which is the unique vector whose length is zero.

Vector Multiplications:

The two different ways to multiply two vectors together: the dot product and the cross product.

The dot product

- The dot product between two vectors is based on the projection of one vector onto another.
- Let's imagine we have two vectors a and b, and we want to calculate how much of a is pointing in the same direction as the vector b.
- We want a quantity that would be positive if the two vectors are pointing in similar directions, zero if they are perpendicular, and negative if the two vectors are pointing in nearly opposite directions. We will define the dot product between the vectors to capture these quantities.
- But first, notice that the question "how much of a is pointing in the same direction as the vector b" does not have anything to do with the magnitude (or length) of b; it is based only on its direction.



- we get a nice symmetric definition for the dot product $a \cdot b$

  $a \cdot b = \|a\| \|b\| \cos\theta$
- the dot product grows linearly with the length of both vectors and is commutative, i.e., $a \cdot b = b \cdot a$

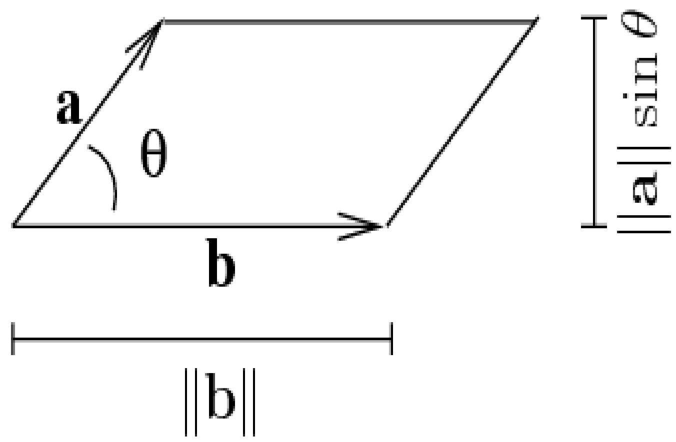
The cross product

- The cross product is defined only for three-dimensional vectors. If **a** and **b** are two three-dimensional vectors, then their cross product, written as **a×b** and pronounced "**a cross b**," is another three-dimensional vector. We define this cross product vector **a×b** by the following three requirements:

1. a×b is a vector that is perpendicular to both a and b.
2. The magnitude (or length) of the vector a×b, written as ‖a×b‖, is the area of the parallelogram spanned by a and b (i.e. the parallelogram whose adjacent sides are the vectors a and b, as shown in below figure).
3. The direction of a×b is determined by the right-hand rule. (This means that if we curl the fingers of the right hand from a to b, then the thumb points in the direction of a×b.)

- we can calculate that the area of the parallelogram spanned by a and b is ‖a‖ ‖b‖ sinθ,

where θ is the angle between a and b. The figure shows the parallelogram as having a base of length ‖b‖ and perpendicular height ‖a‖ sinθ.



- Important properties of the cross product : b×a = −a×b and a×a = 0,

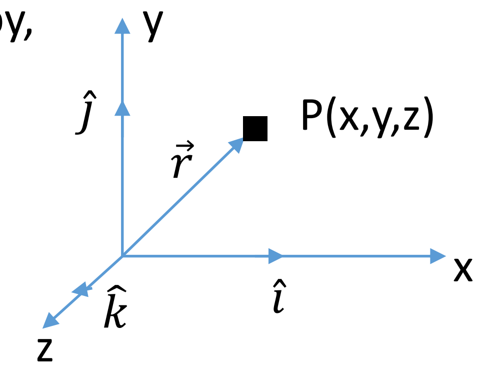
Gradient

- The gradient of a scalar point function Φ is defined as $\nabla\emptyset$ and is written as grad Φ.

$$Grad\ \emptyset = \nabla\emptyset = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\Phi$$

grad Φ Is a vector quantity

- Φ(x, y, z) is a function of three independent variables and its total differential dΦ is given by,



$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

Since $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$

$$d\emptyset = \left[\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right].\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= (\nabla\emptyset).\,d\vec{r}$$

$= |\nabla\emptyset|\ |dr| \cos\theta$

Since θ is the angle between the direction of $\nabla\emptyset$ and dr

If $d\vec{r}$ and $\nabla\emptyset$ are in the same direction, then θ = 0, thus cos θ = 1

Therefore, $d\emptyset = |\nabla\emptyset|\ |dr|$

The value of $d\emptyset$ is greatest when θ = 0

Physical interpretation:

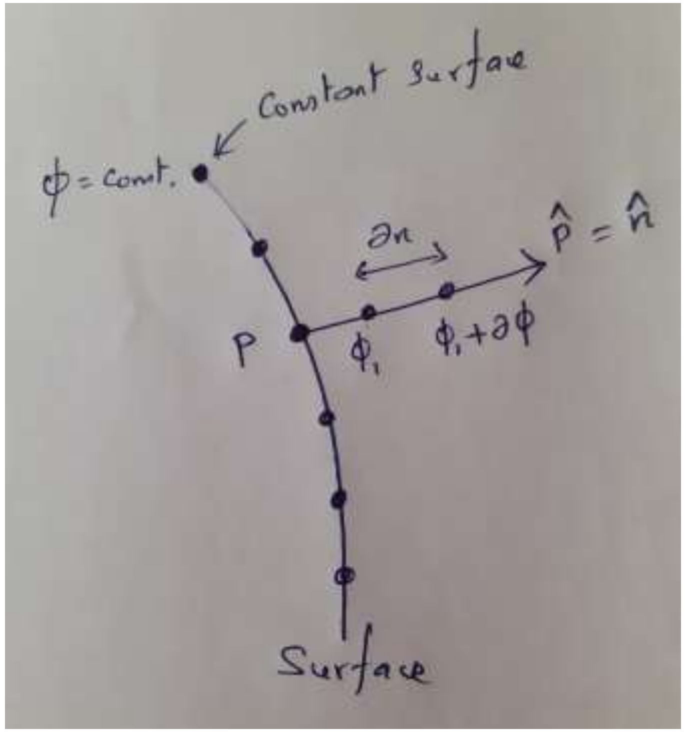
- Gradient of scalar function Φ at any point P is a vector quantity whose magnitude represents the rate of change of Φ with Direction along the normal to the surface defined by the equation Φ(x,y,z) = Constant and its direction is along the normal to the surface.

- $\nabla\emptyset = \left(\frac{\partial\emptyset}{\partial n}\right)\hat{n}$ ⟹ Increment of $\hat{P}$

- Equation of surface Φ(x,y,z) = Constant

  Normal vector to the surface = $\nabla\emptyset$

  Unit normal vector to the surface $(\hat{n})$ = $\frac{\nabla\emptyset}{|\nabla\emptyset|}$



❖ Prove that $\boldsymbol{\nabla r^{n}} = \mathbf{n}\, \boldsymbol{r^{n-2}\, \vec{r}}$

Solution: Position vector, $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$\boldsymbol{\nabla r^{n}} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\boldsymbol{r^{n}}$$

$$= \left[\hat{\imath}\frac{\partial \boldsymbol{r^{n}}}{\partial x} + \hat{\jmath}\frac{\partial \boldsymbol{r^{n}}}{\partial y} + \hat{k}\frac{\partial \boldsymbol{r^{n}}}{\partial z}\right]$$

$$= \left[\hat{\imath}\, n\, \boldsymbol{r^{n-1}}\frac{\partial \boldsymbol{r}}{\partial x} + \hat{\jmath}\, n\, \boldsymbol{r^{n-1}}\frac{\partial \boldsymbol{r}}{\partial y} + \hat{k} n\, \boldsymbol{r^{n-1}}\frac{\partial \boldsymbol{r}}{\partial z}\right]$$

$$= n\, \boldsymbol{r^{n-1}}\left[\hat{\imath}\frac{\partial \boldsymbol{r}}{\partial x} + \hat{\jmath}\frac{\partial \boldsymbol{r}}{\partial y} + \hat{k}\frac{\partial \boldsymbol{r}}{\partial z}\right] \text{----------------(1)}$$

We have, $r^2 = x^2 + y^2 + z^2$
Differentiating partially w.r.t. x, y and z successively , we get

$2r\frac{\partial r}{\partial x}$ = 2x $\quad$ = $\quad \frac{\partial r}{\partial x} = \frac{x}{r}$

$2r\frac{\partial r}{\partial y}$ = 2y $\quad$ = $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$ $\qquad$ -----------------------(2)

$2r\frac{\partial r}{\partial z}$ = 2z $\quad$ = $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

Sub. Equ. (2) in (1), we get,

$$\boldsymbol{\nabla r^{n}} = n\, \boldsymbol{r^{n-1}} \left[\hat{\imath}\ \frac{x}{r} + \hat{\jmath}\ \frac{y}{r} + \hat{k}\ \frac{z}{r}\right]$$

$$= n\, \boldsymbol{r^{n-2}} \left[\hat{\imath}\, x + \hat{\jmath}\, y + \hat{k} z\right]$$

$\boldsymbol{\nabla r^{n}} = n\, \boldsymbol{r^{n-2}}\ \vec{r}$ $\qquad$ Hence proved.

1. **Find the value of $\nabla^2\vec{r}$. Where $\vec{r}$ is position vector.**
2. **If $\vec{r}$ is position vector of a point, deduce the value of grad$\left(\frac{1}{\vec{r}}\right)$**

The Divergence

- The divergence of a vector point function $\vec{F}$ is denoted by **div $\vec{F}$** and is defined by, $\nabla \cdot \vec{F}$

If $\vec{F} = F_1\hat{\imath} + F_2\,\hat{\jmath} + F_3\hat{k}$ then,

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y}\hat{k}\frac{\partial}{\partial z}\right] \cdot \left(F_1\hat{\imath} + F_2\,\hat{\jmath} + F_3\hat{k}\right)$$

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{(scalar value)}$$

- If div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$, then $\vec{F}$ is called **solenoidal vector**.

- The **divergence** of a vector field simply measures how much the flow is expanding at a given point. It **does** not indicate in which direction the expansion is occuring. Hence, the **divergence** is a scalar.

- That is entering fluid in any point which is equal to rate at which fluid is originating per unit volume.

❖ For the position vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, show that

(i) $div\ \hat{r} = 3$

(ii) $div\ \left(\frac{\hat{r}}{r^3}\right) = 0$

(iii) $div(r^n\hat{r}) = (3+n)r^n$

(i) $div\ \hat{r} = \nabla \cdot \hat{r} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$$= \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] = 1 + 1 + 1 = 3$$

(ii) $div\ \left(\frac{\hat{r}}{r^3}\right) = \nabla \cdot \left(\frac{\hat{r}}{r^3}\right) = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r^3}\right)$

$$= \left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$= \frac{1}{r^3} + x\left(-3r^{-4}\frac{\partial r}{\partial x}\right) + \frac{1}{r^3} + y\left(-3r^{-4}\frac{\partial r}{\partial y}\right) + \frac{1}{r^3} + z\left(-3r^{-4}\frac{\partial r}{\partial z}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left[\left(x\frac{\partial r}{\partial x}\right) + \left(y\frac{\partial r}{\partial y}\right) + \left(z\frac{\partial r}{\partial z}\right)\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left[\left(x\frac{x}{r}\right) + \left(y\frac{y}{r}\right) + \left(z\frac{z}{r}\right)\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left(\frac{x^2 + y^2 + z^2}{r}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left(\frac{r^2}{r}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$div\ \left(\frac{\hat{r}}{r^3}\right) = 0$ Hence proved

The Curl

- The curl of a vector point function $\vec{F}$ is denoted by Curl$\vec{F}$ and defined by $\nabla \times \vec{F}$

If $\vec{F} = F_1\hat{\imath} + F_2\,\hat{\jmath} + F_3\hat{k}$ then,

$$\text{Curl}\vec{F} = \nabla \times \vec{F} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y}\hat{k}\frac{\partial}{\partial z}\right] \times \left(F_1\hat{\imath} + F_2\,\hat{\jmath} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad = \text{Vector value}$$

- Any particle rotate in x, y, z axis, it's angular velocity is represented by Curl$\overrightarrow{F}$. Angular velocity $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ direction

- Curl$\vec{F}$ represent the angular velocity at any point of the vector point function. If Curl$\vec{F}$= 0, then there is no rotation is take place

Physical meaning of curl

- We know $V = \omega \times \vec{r}$, where ω is the angular velocity, V is the linear velocity and $\vec{r}$ is the position vector of a point on the rotating body

$$\left[\begin{array}{c}\omega = \omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k} \\ r = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}\end{array}\right]$$

$$curl\, V = \nabla \times V$$

$$= \nabla \times (\omega \times \vec{r})$$

$$= \nabla \times [(\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}) \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})]$$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(\omega_2 z - \omega_3 y)\boldsymbol{i} - (\omega_1 z - \omega_3 x)\boldsymbol{j} + (\omega_1 y - \omega_2 x)\boldsymbol{k}]$$

$$= \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times [(\omega_2 z - \omega_3 y)\boldsymbol{i} - (\omega_1 z - \omega_3 x)\boldsymbol{j} + (\omega_1 y - \omega_2 x)\boldsymbol{k}]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_1 z - \omega_3 x) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\boldsymbol{i} - (-\omega_2 - \omega_2)\boldsymbol{j} + (\omega_3 + \omega_3)\boldsymbol{k}$$

$$= 2(\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k})$$

$$= 2\omega$$

Curl V $= 2\omega$ which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

If Curl **F**$= 0$, the field **F** is termed irrotational

❖For the position vector $\vec{r} = x\hat{\imath}$ + y$\hat{\jmath}$ + z$\hat{k}$, show that

(i) Curl $\hat{r} = 0$

(ii) Curl $\frac{\hat{k}}{r} = \frac{-\hat{\imath}y+\hat{\jmath}x}{r^3}$

(iii) Curl $r^n\hat{r} = 0$

Solution:

(i) Curl $\hat{r}$ = $\nabla \times \hat{r} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

= 0

(ii) Curl $\frac{\hat{k}}{r} = \nabla \times \left(\frac{\hat{k}}{r}\right) = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times \left(\frac{\hat{k}}{r}\right)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r} \end{vmatrix}$$

$$= \hat{\imath}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - \hat{\jmath}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + 0$$

$$= \hat{\imath}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} - \hat{\jmath}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= \hat{\imath}\left(-\frac{1}{r^2}\right)\frac{y}{r} - \hat{\jmath}\left(-\frac{1}{r^2}\right)\frac{x}{r}$$

$$= \frac{-\hat{\imath}y + \hat{\jmath}x}{r^3}$$

Hence proved

The Laplacian

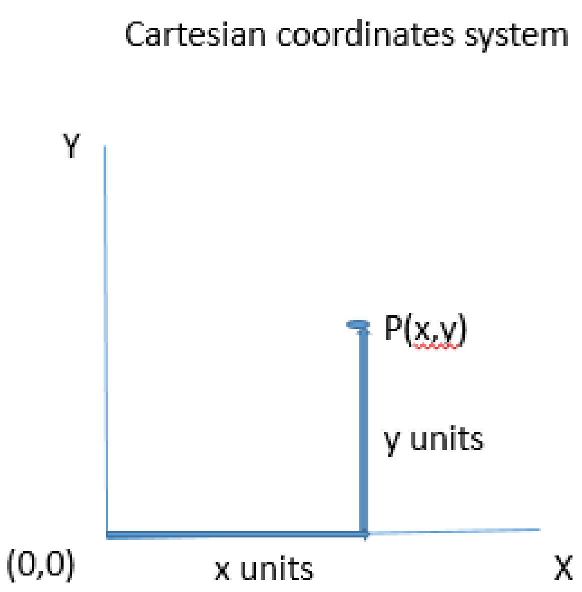
- The divergence of the gradient of a scalar function is called the Laplacian.

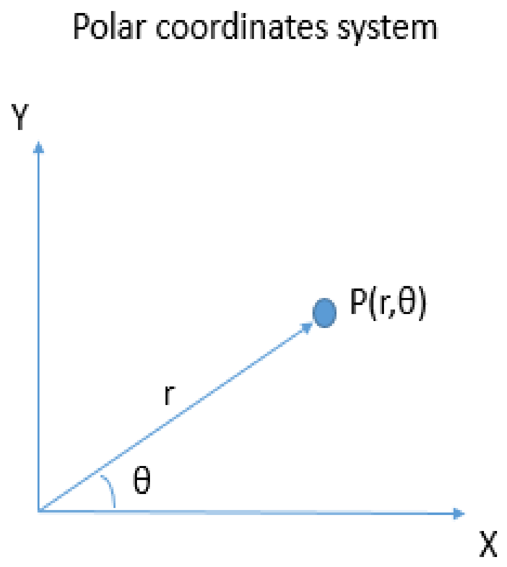
That is, $\nabla \cdot \nabla F = \nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$

- The Laplacian finds application in the Schrodinger equation in Quantum mechanics.

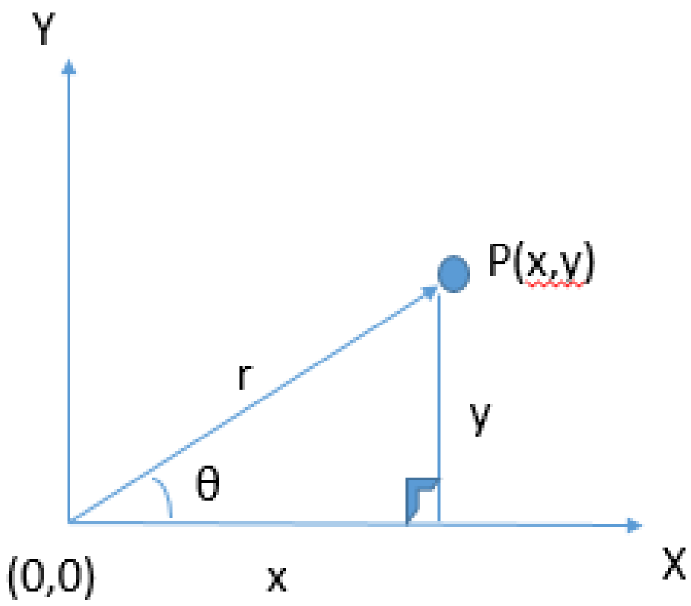
- In electrostatics, it is a part of Laplace's equation and Poisson's equation for relating elective potential to charge density

Orthogonal curvilinear coordinates system





Cartesian coordinates to Polar coordinates



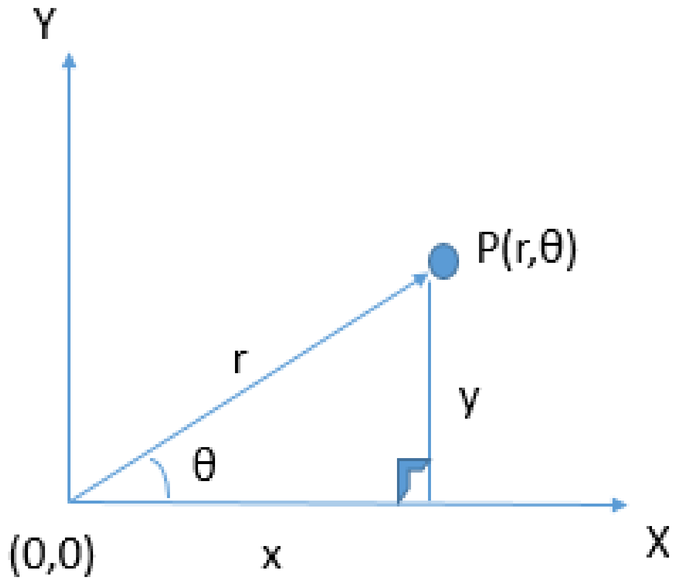
$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$(x, y) \rightarrow \left[\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}\right]$$

Polar coordinate to Cartesian coordinate



$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

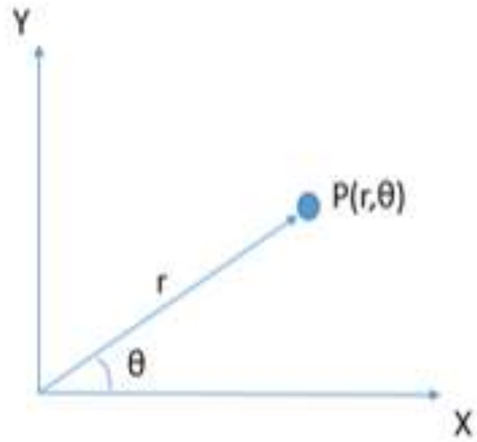
$$(r,\theta) \rightarrow (r\cos\theta, r\sin\theta)$$

Observation

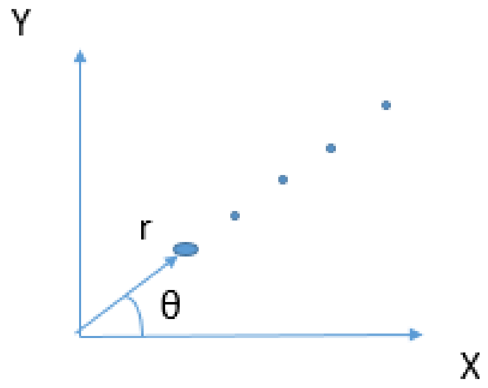
Given : (x, y) ----------- r = f (x, y) and θ = g (x, y)

Given : (r, θ) ----------- x = f (r, θ) and y = g (r, θ)

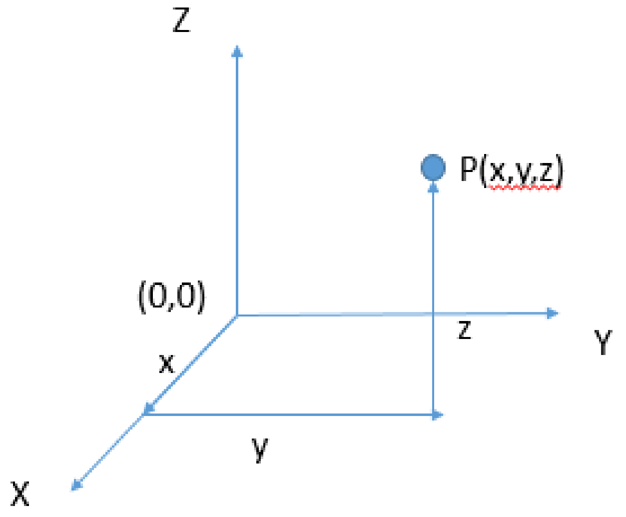
- Case (i) : If r is constant. Θ can vary any value then we will get curve called **θ curve** and it is a **circle**

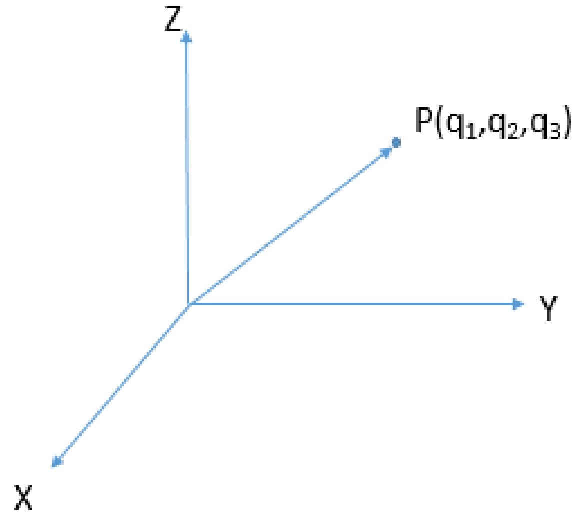


- Case (ii) : If θ is constant and r can vary any value then we will get **r curve**. It is a **straight line**



➢ Why we study the Curvilinear coordinate system ?





Conditions:

$$q_1 = \emptyset_1(x, y, z)$$
$$q_2 = \emptyset_2(x, y, z)$$
$$q_3 = \emptyset_3(x, y, z)$$

q_1, q_2, q_3 are continuous and differentiable

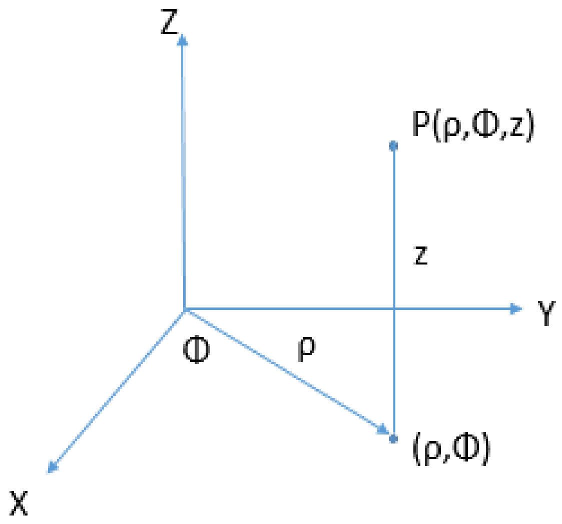
For each point in Cartesian coordinates there will be exactly one point in curvilinear coordinates system

Curvilinear coordinates

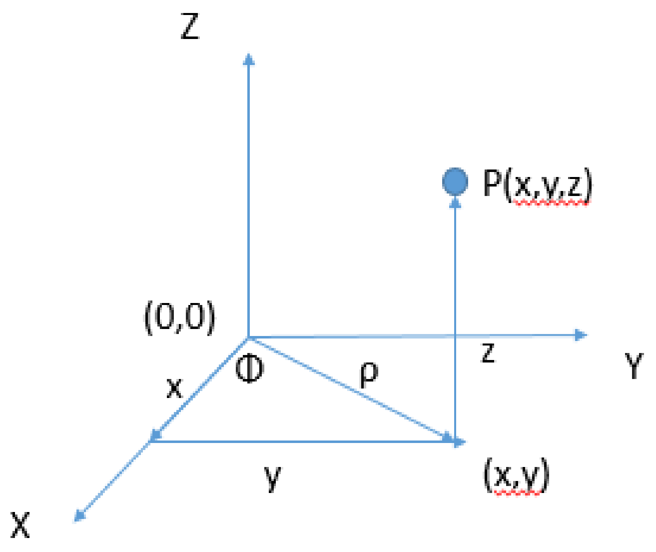
1. Cylindrical coordinates (ρ,Φ,z)
2. Spherical coordinates (r,θ,Φ)

Now how to find **coordinate surfaces** and **coordinate curves** of curvilinear coordinates system

<u>Cylindrical coordinates (ρ,Φ,z)</u>



Cartesian coordinates to Cylindrical coordinates



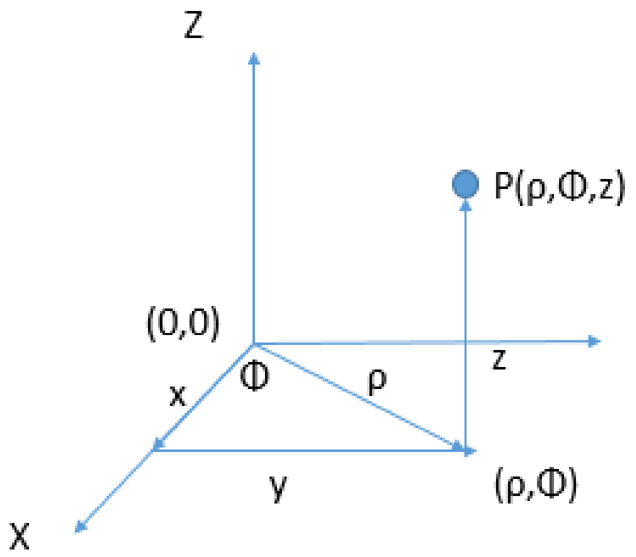
$$\rho = \sqrt{x^2 + y^2}$$

$$\emptyset = \tan^{-1}\frac{y}{x}$$

Z = z

$$(x, y, z) \rightarrow \left(\sqrt{x^2 + y^2},\ \tan^{-1}\frac{y}{x}, z\right)$$

Cylindrical coordinate to Cartesian coordinate system



$$\cos \emptyset = \frac{x}{\rho} \quad \rightarrow \quad x = \rho \cos \emptyset$$

$$\sin \emptyset = \frac{y}{\rho} \quad \rightarrow \quad \rho \sin \emptyset$$

Z = z

(ρ, Φ, z) → $(\rho \cos \emptyset , \rho \sin \emptyset, z)$

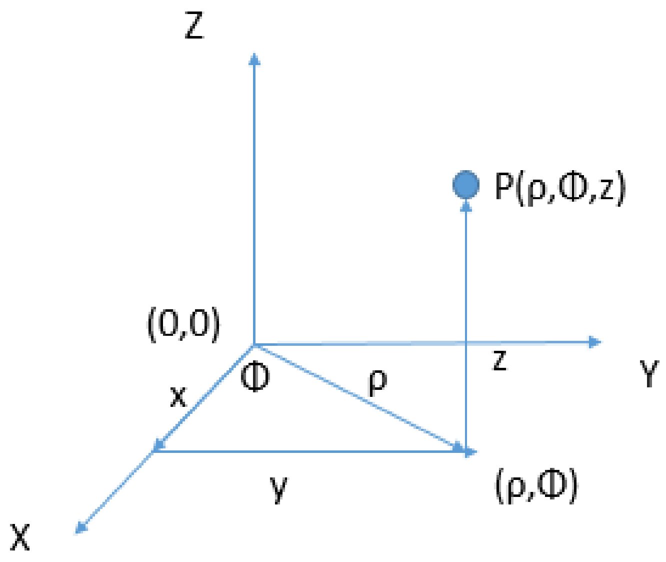
Therefore the values → $\rho \geq 0,\ 0 \leq \emptyset \leq 2\pi,$ z $\in (-\infty , \infty)$

Coordinate Surfaces:

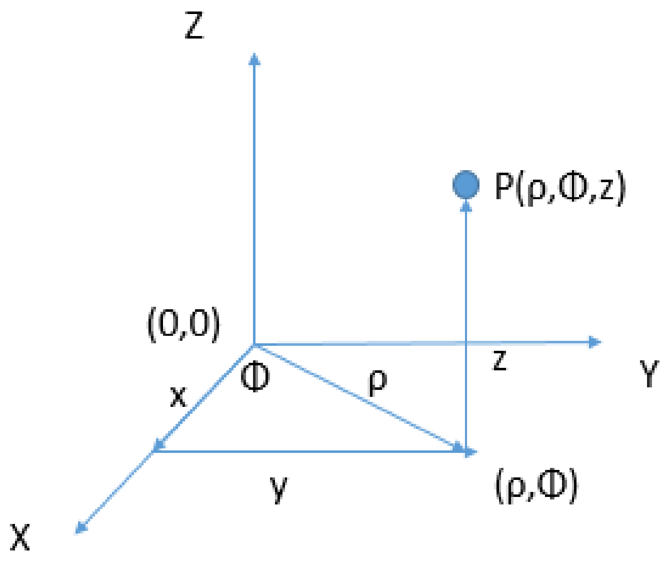
- If ρ = constant and Φ, z can take any value then it will be a **Cylinder co-oxial with z-axis**

- If Φ = constant and ρ, z can take any value then it will be a **Plane through z-axis**

- If z = constant and ρ, Φ can take any value then it will be **a Plane perpendicular to z-axis**



Coordinate Curves:

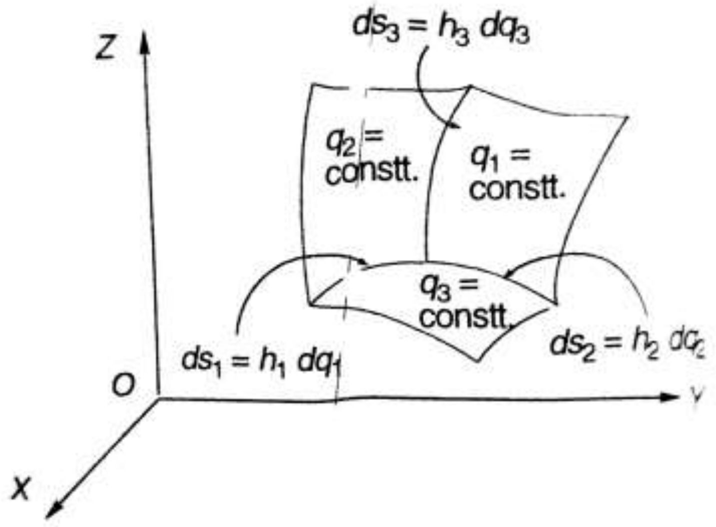


- If ρ, Φ = constant and z can take any value, it will be a **Straight line parallel to z- axis (z- curve)**

- If ρ, z = constant and Φ can take any value, it will be a **Circle (Φ - curve)**

- If Φ, z = constant and ρ can take any value, it will be a **Straight line (ρ - curve)**

Orthogonal Curvilinear Coordinates



- In Cartesian coordinates the position of a point P(x, y, z) is determined by the intersection of three mutually perpendicular planes x= constant, y= constant and z= constant

- Let the three new families of surfaces, described by q_1 = constant, q_2= constant, q_3= constant, intersect at point P.

- The values of q_1, q_2, q_3 for the three surfaces intersecting at P are called the **curvilinear coordinates** of P. The three new surfaces are then called **the coordinate surfaces** or **curvilinear surfaces**

- If the coordinates surfaces are mutually perpendicular at every point **P(x, y, z),** then the curvilinear coordinates **(q_1, q_2, q_3)** are said to be **orthogonal curvilinear coordinates**

- The coordinate surfaces intersect pairwise in three curves, called the **coordinate lines** or **coordinate curves**

- The coordinate axes are determined by the tangents to the coordinate lines at the intersection of the three surfaces

- Obviously any given point P may be identified by curvilinear coordinates **(q_1, q_2, q_3)** as well as by Cartesian coordinates **(x, y, z)**. This means that in principle we may write

$$x = x\,(q_1, q_2, q_3)$$
$$y = y\,(q_1, q_2, q_3) \qquad \text{------------------------(1)}$$
$$z = z\,(q_1, q_2, q_3)$$

With inverses

$$q_1 = q_1\,(x, y, z)$$
$$q_2 = q_2\,(x, y, z) \qquad \text{------------------------(2)}$$
$$q_3 = q_3\,(x, y, z)$$

- With each family of surface q_i = constant, we can associate a unit vector $\widehat{u_i}$ normal to each surface q_i = constant and in the direction of increasing $q_{\dot{r}}$

The partial differentiation of equation (1) yields

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3 \qquad \text{-------------------------(3)}$$

$$dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3$$

Hence the square of the distance between two neighbouring points is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{ij} {h_{ij}}^2 dq_i dq_j \qquad \text{(i, j = 1, 2, 3)} \qquad \text{---------------------(4)}$$

Where the coefficients ${h_{ij}}^2$ are given by

$${h_{ij}}^2 = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} \qquad \text{-----------------------(5)}$$

The most useful coordinate systems are orthogonal ones. i.e., the systems in which surfaces always intersect at right angles. At this point we limit ourselves to orthogonal coordinate systems which means

$$h_{ij} = 0, i \neq j \qquad \text{------------------(6)}$$

Now to simplify the notation, we write $h_{ij} = h_i$ so that equation (4) yields,

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2 \qquad \text{-------------------(7)}$$

The distance between any two points on a coordinate line is called the **line element**. When the variation is limited to any given q_i holding the other q’s constant, then the line element, form equation (4), is given by

$$ds_i = h_i dq_i \qquad \text{---------------------- (8)}$$

From this equation it may be noted that three curvilinear coordinates q_1, q_2, q_3 need not be lengths. The scale factors h_i may depend on q’s and they may have dimensions. The product $\boldsymbol{h_i dq_i}$ must have the dimensions of length

From equation (8) we may develop the surface and volume elements. The three possible surface elements in orthogonal systems thus become

$$dS_{ij} = ds_i ds_j = h_i h_j dq_i dq_j \qquad (i, j = 1,2,3; i \neq j) \text{ -------------- } (9)$$

and the volume element

$$d\tau = ds_1 ds_2 ds_3 = h_1 h_2 h_3 \, dq_1 dq_2 dq_3 \text{ ----------- } (10)$$

From the above equation, $h_1 h_2 h_3$ are scaling factor and $dq_1 dq_2 dq_3$ are differential of the elements

Differential operators in terms of orthogonal curvilinear coordinates

- Consider three mutually perpendicular coordinate surfaces described by q_1 = constant, q_2= constant, q_3= constant.

- Let $\Psi(\mathbf{q_1, q_2, q_3})$ be a scalar function and **V** be a vector function with components V_1, V_2, V_3 in the three directions in which $\mathbf{q_1, q_2, q_3}$ increase.

- If $\widehat{u_1}, \widehat{u_2}, \widehat{u_3}$ are unit vectors along the directions of increasing $\mathbf{q_1, q_2, q_3}$ respectively, then vector V in terms of orthogonal curvilinear coordinates may be written as

$$\mathbf{V} = \widehat{u_1} V_1 + \widehat{u_2} V_2 + \widehat{u_3} V_3$$

The Gradient

- The gradient of a scalar function Ψ is a vector whose magnitude and direction give the maximum space rate of change of scalar function Ψ . From this interpretation the component of $\nabla \Psi(\mathbf{q_1, q_2, q_3})$ in the direction normal to the surface $\mathbf{q_1}$ **= constant** and hence in the direction of q_1 is

$$[\nabla\,\Psi]_1 = \lim_{\delta s1\to 0} \frac{\delta\Psi}{\delta s1} = \frac{\partial\Psi}{\partial s1} = \frac{\partial\Psi}{h_1\partial q_1} = \frac{1}{h_1}\frac{\partial\Psi}{\partial q_1} \qquad ----------- -(1)$$

Where δs_1=$h_1\partial q_1$is the differential length in the direction of increasing q_1 and $\partial\Psi$ represents an increase in Ψ on travelling a distance δs_1 in the limit $\delta s_1 \rightarrow 0$.

By repeating equation (1) for q_2 and q_3 , we get

$$[\nabla\,\Psi]_2 = \frac{1}{h_2}\frac{\partial\Psi}{\partial q_2} \qquad -------- -(2)$$

$$[\nabla\,\Psi]_3 = \frac{1}{h_3}\frac{\partial\Psi}{\partial q_3} \qquad -------- -(3)$$

Adding equations (1), (2) and (3) vectorially, the gradient of scalar function Ψ in orthogonal curvilinear coordinates becomes

$$\text{grad } \Psi \quad = \quad \nabla\Psi \quad = \frac{\widehat{u_1}}{h_1}\frac{\partial\Psi}{\partial q_1} + \frac{\widehat{u_2}}{h_2}\frac{\partial\Psi}{\partial q_2} + \frac{\widehat{u_3}}{h_3}\frac{\partial\Psi}{\partial q_3}$$

Thus the operator grad in orthogonal curvilinear coordinates is,

$$\text{grad } \Psi \quad = \quad \nabla\Psi \quad = \left[\frac{\widehat{u_1}}{h_1}\frac{\partial}{\partial q_1} + \frac{\widehat{u_2}}{h_2}\frac{\partial}{\partial q_2} + \frac{\widehat{u_3}}{h_3}\frac{\partial}{\partial q_3}\right]\Psi$$

In Cartesian coordinates,

$$Grad\ \Psi = \nabla\Psi = \left[\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right]\Psi$$

The divergence

The divergence in orthogonal curvilinear coordinates can be written as,

$$\text{div } \mathbf{V} = \nabla\ .\mathbf{V} \ = \ \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(V_1 h_2 h_3)}{\partial q_1} + \frac{\partial(V_2 h_3 h_1)}{\partial q_2} + \frac{\partial(V_3 h_1 h_2)}{\partial q_3}\right]$$

In Cartesian coordinates,

$$\text{div } \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

The curl

The curl in orthogonal curvilinear coordinates can be written as,

$$\text{Curl}\,\vec{V} = \nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\widehat{u_1} & h_2\widehat{u_2} & h_3\widehat{u_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1 h_1 & V_2 h_2 & V_3 h_3 \end{vmatrix}$$

In Cartesian coordinates,

$$\text{Curl}\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

The Laplacian

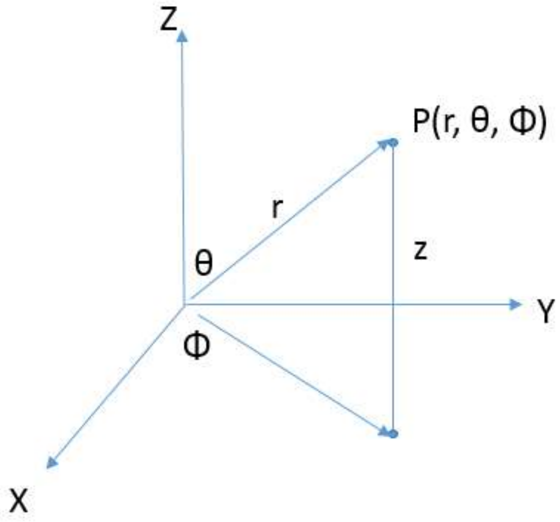
In Cartesian coordinates,

$$\nabla \cdot \nabla\Psi = \nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$$

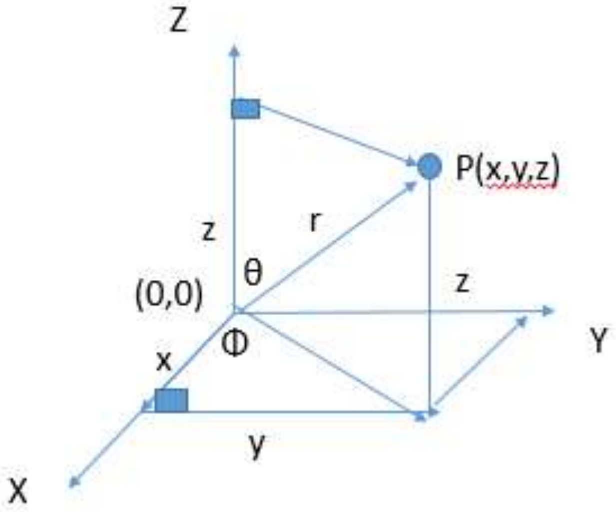
The Laplacian may be obtained be combining gradient and divergence

$$\nabla^2\Psi = \nabla \cdot \nabla\Psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\Psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\Psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\Psi}{\partial q_3}\right)\right]$$

Spherical polar coordinates



Cartesian to Spherical coordinates



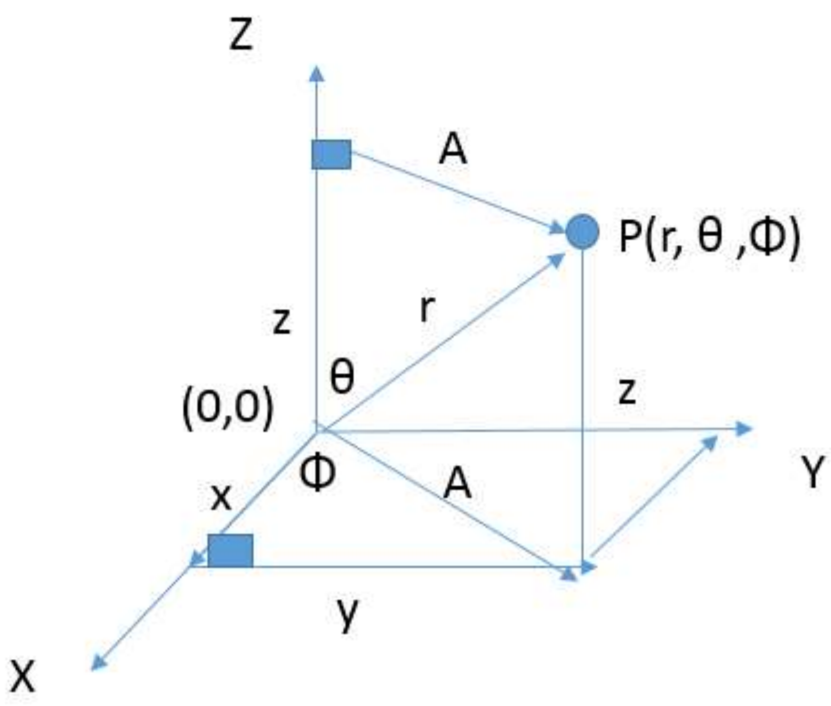
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \emptyset = \frac{y}{x} \rightarrow \quad \emptyset = \tan^{-1} {}^{y}\!/_{x}$$

$$\cos \theta = \frac{z}{r} \rightarrow \; \theta = \cos^{-1} {}^{z}\!/_{r}$$

$$(x, y, z) \rightarrow (\sqrt{x^2 + y^2 + z^2}, \tan^{-1} {}^{y}\!/_{x}\, , \cos^{-1} {}^{z}\!/_{r}\,)$$

Spherical to Cartesian coordinates



$$\sin\emptyset = \frac{y}{A} \quad \rightarrow \quad y = A\sin\emptyset$$

$$\cos\emptyset = \frac{x}{A} \quad \rightarrow \quad x = A\cos\emptyset$$

$$\cos\theta = \frac{z}{r} \quad \rightarrow \quad z = r\cos\theta$$

$$\sin\theta = \frac{A}{r} \quad \rightarrow \quad A = r\sin\theta$$

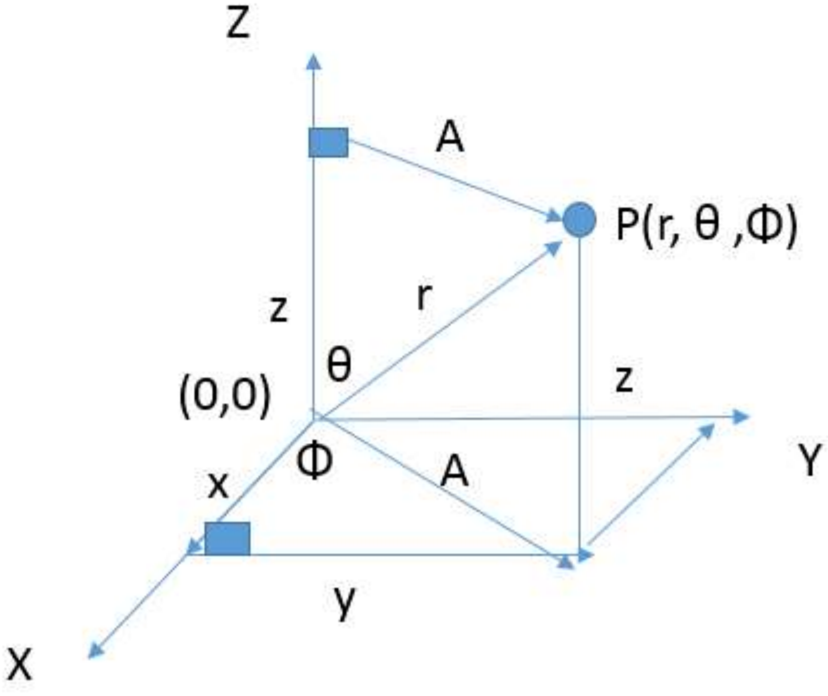
Therefore, $x = r\sin\theta\cos\emptyset$

$$y = r\sin\theta\sin\emptyset$$

$$z = r\cos\theta$$

(r,Φ, θ) → $(r\sin\theta\cos\emptyset\,, r\sin\theta\sin\emptyset\,, r\cos\theta)$

Spherical polar coordinates (r, θ ,Φ) and differential operators:



The spherical polar coordinate system consists of

- Concentric spheres about the origin O , $r = \sqrt{x^2 + y^2 + z^2}$
- Right circular cones about Z-axis with the vertices at the origin O, $\cos\theta = \frac{z}{r} \rightarrow \ \theta = \cos^{-1} z/r$
- Half planes through the Z-axis, $\tan\emptyset = \frac{y}{x} \rightarrow \ \ \emptyset = \tan^{-1} y/x$

The transformation between rectangular coordinates (x, y, z) and spherical coordinates (r, θ ,Φ) are given by

$$x = r\sin\theta\cos\emptyset$$

$$y = r\sin\theta\sin\emptyset \qquad ----------(1)$$

$$z = r\cos\theta$$

We have, $$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \emptyset} d\emptyset$$

$$dx = \sin\theta \cos\emptyset \, dr + r\cos\theta\cos\emptyset \, d\theta - r\sin\theta\sin\emptyset \, d\emptyset \quad \text{----------------}(2)$$

Similarly, $$dy = \sin\theta\sin\emptyset \, dr + r\cos\theta\sin\emptyset \, d\theta + r\sin\theta\cos\emptyset \, d\emptyset \qquad ------(3)$$

$$dz = \cos\theta \, dr - r\sin\theta \, d\theta \qquad -------(4)$$

We know that the line element ds in Cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad --------(5)$$

Substituting equations (2), (3) and (4) in equation (5) , the expression for the line element in spherical polar coordinates becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\emptyset^2 \qquad ----------(6)$$

We know that, $ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$ -------------------(7)

Comparing equ.(6) with (7), we get

$h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$

$q_1 = r$, $q_2 = \theta$, $q_3 = \Phi$ -----------------------(8)

Using this, now we can write differential operators in spherical polar coordinates

Differential operators in spherical polar coordinates

Gradient :

In orthogonal curvilinear coordinates grad Ψ is

$$\text{grad}\,\Psi \quad = \quad \nabla\Psi \quad = \frac{\widehat{u_1}}{h_1}\frac{\partial\Psi}{\partial q_1} + \frac{\widehat{u_2}}{h_2}\frac{\partial\Psi}{\partial q_2} + \frac{\widehat{u_3}}{h_3}\frac{\partial\Psi}{\partial q_3}$$

If $\widehat{u_r}, \widehat{u_\theta}, \widehat{u_\emptyset}$ are unit vectors along r, θ ,Φ axes respectively, then using equation (8) , grad Ψi in spherical polar coordinates may expressed as

$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta$

$q_1 = r, \quad q_2 = \theta, \quad q_3 = \Phi$

$$\text{grad}\,\Psi \quad = \frac{\partial\Psi}{\partial r}\widehat{u_r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\widehat{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\emptyset}\widehat{u_\emptyset}$$

Or

$$\text{grad} \quad = \quad \nabla \quad = \frac{\partial}{\partial r}\widehat{u_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\widehat{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\emptyset}\widehat{u_\emptyset}$$

Divergence :

In orthogonal curvilinear coordinates div **V** is

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(V_1 h_2 h_3)}{\partial q_1} + \frac{\partial(V_2 h_3 h_1)}{\partial q_2} + \frac{\partial(V_3 h_1 h_2)}{\partial q_3}\right]$$

$h_1 = 1,\quad h_2 = r,\quad h_3 = r \sin\theta$

Using equation (8) , div **V** in spherical polar coordinates may expressed as

$q_1 = r,\quad q_2 = \theta,\quad q_3 = \Phi$

$$div\ \boldsymbol{V} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\ V_r) + \frac{\partial}{\partial\theta}(r \sin\theta\ V_\theta) + \frac{\partial}{\partial\emptyset}(r\ V_\emptyset)\right]$$

$$div\ \boldsymbol{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\ V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\emptyset}{\partial\emptyset}$$

Curl : In orthogonal curvilinear coordinates Curl **V** is

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \widehat{u_1} & h_2 \widehat{u_2} & h_3 \widehat{u_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1 h_1 & V_2 h_2 & V_3 h_3 \end{vmatrix}$$

Using equation (8) , Curl **V** in spherical polar coordinates may expressed as

$h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$

$q_1 = r$, $q_2 = \theta$, $q_3 = \Phi$

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \widehat{u_r} & r\widehat{u_\theta} & r \sin \theta \, \widehat{u_\emptyset} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \emptyset} \\ V_r & rV_\theta & r \sin \theta \, V_\emptyset \end{vmatrix}$$

$$\text{Curl } \vec{V} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta \, V_\emptyset) - \frac{\partial V_\theta}{\partial \emptyset} \right] \widehat{u_r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial V_r}{\partial \emptyset} - \frac{\partial (r V_\emptyset)}{\partial r} \right] \widehat{u_\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r V_\theta) - \frac{\partial V_r}{\partial \theta} \right] \widehat{u_\emptyset}$$

Laplacian :

In orthogonal curvilinear coordinates $\nabla^2\Psi$ is

$$\nabla^2\Psi = \nabla\cdot\nabla\Psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Psi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial q_3}\right)\right]$$

Using equation (8) , $\nabla^2\Psi$ in spherical polar coordinates may expressed as

$h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$

$q_1 = r$, $q_2 = \theta$, $q_3 = \Phi$

$$\nabla^2\Psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\phi}\right)\right]$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

Cylindrical coordinates and differential operators :

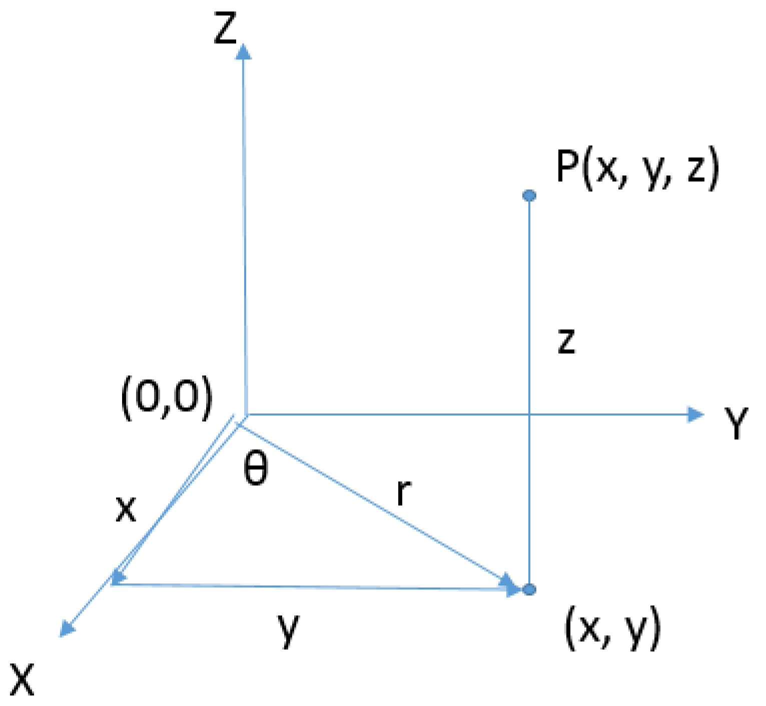
The cylindrical coordinate system consists of :

- Right circular cylinders having Z-axis as common, which form families of concentric circles about the origin O in X-Y plane

$$r = \sqrt{x^2 + y^2}$$

- Half planes through to Z-axis, $\theta = \tan^{-1}\frac{y}{x}$

- Planes parallel to X-Y plane, Z = z



Thus the position of point P in cylindrical coordinates is specified by (r, θ ,z) where ‘r’ is the distance in the X-Y plane from the origin to the cylinder on which the point P lies, ‘θ’ is angle makes from positive X-axis in X-Y plane and ‘z’ is the distance from the X-Y plane to the point P

From Figure, the transformations between rectangular coordinates (x,y,z) and cylindrical coordinates (r, θ ,z) are given by

$$\cos\theta = \frac{x}{r} \rightarrow \ x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \rightarrow \ y = \ r\sin\theta \quad \text{-------(1)}$$

Z = z

We have, $$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$

Therefore from equ (1)

$$\mathrm{dx} = \cos\theta\ dr - r\sin\theta\ d\theta$$

Similarly, $$dy = \sin\theta\ dr + r\cos\theta\ d\theta \qquad and \quad dZ = dz \qquad ----\,-(2)$$

We know that the line element ds in Cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad\qquad --------\,-(3)$$

Substituting equation (2) in equation (3) , the expression for the line element in cylindrical coordinates becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \quad \text{----------------- (4)}$$

We know that, $$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2 \quad \text{-------------------(5)}$$

Comparing equ. (4) with (5), we get

$h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

$q_1 = r, \quad q_2 = \theta, \quad q_3 = z$ ------------------(6)

Now we shall write differential operators in cylindrical coordinates

differential operators :

Gradient : In orthogonal curvilinear coordinates grad Ψ is

$$\text{grad } \Psi \quad = \quad \nabla\Psi \quad = \frac{\widehat{u_1}}{h_1}\frac{\partial\Psi}{\partial q_1} + \frac{\widehat{u_2}}{h_2}\frac{\partial\Psi}{\partial q_2} + \frac{\widehat{u_3}}{h_3}\frac{\partial\Psi}{\partial q_3}$$

If $\widehat{u_r}, \widehat{u_\theta}, \widehat{u_z}$ are unit vectors along r, θ ,z axes respectively, then using equation (6) , grad Ψi in cylindrical coordinates may expressed as

$h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

$$\text{grad } \Psi \quad = \frac{\partial\Psi}{\partial r}\widehat{u_r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\widehat{u_\theta} + \frac{\partial\Psi}{\partial\emptyset}\widehat{u_z}$$

$q_1 = r, \quad q_2 = \theta, \quad q_3 = z$

Divergence :

In orthogonal curvilinear coordinates div **V** is

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(V_1 h_2 h_3)}{\partial q_1} + \frac{\partial(V_2 h_3 h_1)}{\partial q_2} + \frac{\partial(V_3 h_1 h_2)}{\partial q_3}\right]$$

Using equation (6) , div **V** in cylindrical coordinates may expressed as

$h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

$q_1 = r, \quad q_2 = \theta, \quad q_3 = z$

$$div\,\boldsymbol{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\,V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

Curl :

In orthogonal curvilinear coordinates Curl **V** is

$$\text{Curl}\ \vec{V} = \nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\widehat{u_1} & h_2\widehat{u_2} & h_3\widehat{u_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1 h_1 & V_2 h_2 & V_3 h_3 \end{vmatrix}$$

Using equation (6) , Curl **V** in cylindrical coordinates may expressed as

$h_1 = 1$, $h_2 = r$, $h_3 = 1$

$q_1 = r$, $q_2 = \theta$, $q_3 = z$

$$\text{Curl}\ \vec{V} = \nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} \widehat{u_r} & r\widehat{u_\theta} & \widehat{u_z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

$$\text{Curl}\ \vec{V} = \frac{1}{r}\left[\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right]\widehat{u_r} + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\widehat{u_\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta}\right]\widehat{u_z}$$

Laplacian :

In orthogonal curvilinear coordinates $\nabla^2\Psi$ is

$$\nabla^2\Psi = \nabla\cdot\nabla\Psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Psi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial q_3}\right)\right]$$

Using equation (6) , $\nabla^2\Psi$ in cylindrical coordinates may expressed as

$h_1 = 1$, $h_2 = r$, $h_3 = 1$

$q_1 = r$, $q_2 = \theta$, $q_3 = z$

$$\nabla^2\Psi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial\Psi}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\Psi}{\partial z}\right)\right]$$

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\partial^2\Psi}{\partial z^2}$$